

Appendix: Analytical thresholds reveal mechanisms of cross-scale dynamics in adaptive mutualistic networks

Fernanda S. Valdovinos^{1,2}

Appendix to the Research Article “Analytical thresholds reveal mechanisms of cross-scale dynamics in adaptive mutualistic networks.”

A note on the degree-scaled reward budget

For any persisting pollinator j , setting $\frac{da_j}{dt} = 0$ in the pollinator dynamics yields the general zero-growth condition

$$\bar{R}_j^* \equiv \sum_{i \in \mathcal{P}_j} \alpha_{ij}^* R_i^* = R^c, \quad R^c \equiv \frac{\mu^A}{cb\tau}, \quad (\text{A1})$$

where $\bar{R}_j^*$ is the effort-weighted mean reward experienced by pollinator j at equilibrium, and R^c is the effort-weighted reward required for zero per-capita growth. The corresponding unweighted equilibrium reward budget, denoted by $\mathcal{R}_j^*$, is evaluated over the diet relevant to the specified foraging rule.

Without adaptive foraging. Foraging effort is distributed uniformly, $\alpha_{ij} = 1/d_j$ for all $i \in \mathcal{P}_j$, where $d_j = |\mathcal{P}_j|$. Substituting this fixed allocation into Eq. (A1) gives

$$\frac{1}{d_j} \sum_{i \in \mathcal{P}_j} R_i^* = R^c \implies \mathcal{R}_j^* \equiv \sum_{i \in \mathcal{P}_j} R_i^* = R^c d_j. \quad (\text{A2})$$

Thus, under uniform fixed foraging, zero growth requires the arithmetic mean reward across the complete binary diet to equal R^c , or equivalently for the unweighted equilibrium reward budget across that diet to equal $R^c d_j$.

With adaptive foraging. Under adaptive foraging, effort is redistributed until rewards equalize across plants receiving positive equilibrium effort. Therefore, for every plant in pollinator j 's active equilibrium diet $\mathcal{P}_j^+$,

$$R_i^* = R^* = R^c, \quad i \in \mathcal{P}_j^+. \quad (\text{A3})$$

Summing the common reward over the active equilibrium diet gives

$$\mathcal{R}_j^* \equiv \sum_{i \in \mathcal{P}_j^+} R_i^* = \sum_{i \in \mathcal{P}_j^+} R^* = R^* d_j^+ = R^c d_j^+, \quad (\text{A4})$$

where $d_j^+ = |\mathcal{P}_j^+| \leq d_j$, with equality only when all binary links receive positive equilibrium effort. Links present in the binary network but receiving zero equilibrium effort do not enter this adaptive-foraging identity.

Interpretation. Eqs. (A2)–(A4) show that the unweighted equilibrium reward budget scales linearly with the number of plants over which the relevant foraging rule distributes positive effort. Under uniform fixed foraging, this set is the complete binary diet, and $\mathcal{R}_j^* = R^c d_j$. Under adaptive foraging, it is the active equilibrium diet, and $\mathcal{R}_j^* = R^c d_j^+$. If all binary links remain active, then $d_j^+ = d_j$, and the two identities have the same numerical form.

These degree-scaled expressions are reward-budget identities, not independent physiological thresholds. In both foraging regimes, pollinator growth is governed by the effort-weighted mean reward $\bar{R}_j$: positive growth requires $\bar{R}_j > R^c$, whereas a persisting pollinator at equilibrium satisfies $\bar{R}_j = R^c$. The two regimes differ in how this condition is attained. Under uniform fixed foraging, it constrains the arithmetic mean reward across the complete binary diet. Under adaptive foraging, it follows from reward equalization across plants receiving positive equilibrium effort. The consequences of this difference for specialist and generalist persistence in nested networks are developed in the main text.

Table A1: Model notation, parameters, and derived quantities. $\mathcal{P}_j$ and $\mathcal{A}_i$ are the sets of plants connected to pollinator j and pollinators connected to plant i , respectively, and $d_j = |\mathcal{P}_j|$. Under adaptive foraging, $\mathcal{P}_j^+ = \{i \in \mathcal{P}_j : \alpha_{ij}^* > 0\}$ and $d_j^+ = |\mathcal{P}_j^+|$ are the active equilibrium diet and degree. The main text retains τ where it enters $c' = cb\tau$ but otherwise fixes its numerical value at 1 and suppresses it from expanded expressions; it is retained here to show the corresponding dimensionally explicit forms.

Definition	Symbol	Dimension
<i>State variables</i>		
Plant density	p_i	ind. area ⁻¹
Pollinator density	a_j	ind. area ⁻¹
Standing floral reward on plant i	R_i	mass area ⁻¹
Foraging effort allocated by pollinator j to plant i	α_{ij}	—
<i>Functions</i>		
Visitation rate from pollinator j to plant i	$V_{ij} = \alpha_{ij} \tau a_j p_i$	visits area ⁻¹ time ⁻¹
Visit quality from pollinator j to plant i	$\sigma_{ij} = V_{ij} / \sum_{k \in \mathcal{P}_j} V_{kj}$	—
Germination–competition factor for plant i	$\gamma_i = g_i (1 - \sum_{l \neq i} u_l p_l - w_i p_i)$	—
<i>Parameters</i>		
Visitation efficiency	τ	visits area time ⁻¹ ind. ⁻²
Seeds produced per pollination event	e_i	ind. visits ⁻¹
Plant mortality rate	μ_i^P	time ⁻¹
Pollinator mortality rate	μ^A	time ⁻¹
Reward-to-pollinator conversion rate	c	ind. mass ⁻¹
Reward extraction efficiency	b	ind. visits ⁻¹
Effective reward-to-growth conversion	$c' = cb\tau$	area time ⁻¹ mass ⁻¹
Reward production rate	β_i	mass ind. ⁻¹ time ⁻¹
Reward self-limitation rate	ϕ_i	time ⁻¹
Maximum germination fraction	g_i	—
Interspecific competition coefficient	u_l	area ind. ⁻¹
Intraspecific competition coefficient	w_i	area ind. ⁻¹
Foraging adaptation rate	G	—
<i>Derived quantities used in the analysis</i>		
Effort-weighted mean reward experienced by pollinator j	$\bar{R}_j = \sum_{i \in \mathcal{P}_j} \alpha_{ij} R_i$	mass area ⁻¹
Pollinator zero-growth reward threshold	$R^c = \mu^A / c' = \mu^A / (cb\tau)$	mass area ⁻¹
Common reward across the active equilibrium diet under adaptive foraging	$R^* = R^c$	mass area ⁻¹
Active equilibrium diet under adaptive foraging	$\mathcal{P}_j^+ = \{i \in \mathcal{P}_j : \alpha_{ij}^* > 0\}$	—
Active degree under adaptive foraging	$d_j^+ = \mathcal{P}_j^+ \leq d_j$	—
Equilibrium reward budget under uniform fixed foraging	$\mathcal{R}_j^* = \sum_{i \in \mathcal{P}_j} R_i^* = R^c d_j$	mass area ⁻¹
Equilibrium reward budget under adaptive foraging	$\mathcal{R}_j^* = \sum_{i \in \mathcal{P}_j^+} R_i^* = R^* d_j^+ = R^c d_j^+$	mass area ⁻¹
Effort-weighted pollinator abundance on plant i	$v_i = \sum_{j \in \mathcal{A}_i} \alpha_{ij} a_j$	ind. area ⁻¹
Equilibrium abundance-weighted effort sustained by plant i	$v_i^* = (\beta_i p_i^* - \phi_i R^*) / (b\tau R^*)$	ind. area ⁻¹
Pollination service received by plant i	$Q_i = \sum_{j \in \mathcal{A}_i} \sigma_{ij} \alpha_{ij} a_j$	ind. area ⁻¹
Plant pollination threshold	$Q_i^c = \mu_i^P / (g_i e_i \tau U_i^*)$	ind. area ⁻¹
Mean visit quality received by plant i	$\bar{\sigma}_i^* = Q_i^* / v_i^*$	—
Remaining recruitment niche for plant i	$U_i^* = 1 - \sum_{l \neq i} u_l p_l^*$	—

¹ Department of Environmental Science and Policy, University of California, Davis, California, USA.

² Santa Fe Institute, Santa Fe, New Mexico, USA.

Corresponding author: fvaldovinos@ucdavis.edu